On the Schläfli differential formula

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To F. O. Almgren, in memoriam

Abstract

The celebrated formula of Schläfli relates the variation of the dihedral angles of a smooth family of polyhedra in a space form and the variation of volume. We give a smooth analogue of this classical formula – our result relates the variation of the volume bounded by a hypersurface moving in a general Einstein manifold and the integral of the variation of the mean curvature. The argument is direct, and the classical polyhedral result (as well as results for Lorenzian space forms) is an easy corollary. We extend it to variations of the metric in a Riemannian Einstein manifold with boundary.

We apply our results to extend the classical Euclidean inequalities of Aleksandrov to other 3-dimensional constant curvature spaces. We also obtain rigidity results for Ricci-flat manifolds with umbilic boundaries and existence results for foliations of Einstein manifolds by hypersurfaces.

Résumé

La formule classique de Schläfli relie la variation des angles dièdres d'une famille lisse de polyèdres dans un espace à courbure constante et la variation du volume.

On donne un analogue régulier de cette formule classique – notre resultat relie la variation du volume borné par une hypersurface se déplaçant dans une variété d'Einstein à l'intégrale de la variation de la courbure moyenne. Puis nous l'étendons aux variations de la métrique à l'intérieur d'une variété d'Einstein riemannienne.

Comme application, on donne un résultat de rigidité pour les variétés Ricci-plates à bord ombilique, quelques formules concernant les feuilletages de variétés d'Einstein par des hypersurfaces, ainsi que des analogues de la formule d'Alexandrov-Fenchel dans les variétés de dimension 3 à courbure constante.

Let M be a Riemannian $(m + 1)$-dimensional space-form with constant curvature K, and $(P_t)_{t \in [0,1]}$ a one-parameter family of polyhedra in M bounding compact domains, all having the same combinatorics. Call V_t the volume bounded by P_t, $\theta_{i,t}$ and $W_{i,t}$ the dihedral angle and the $(m - 1)$-volume of the codimension 2 face i of P_t. The classical Schläfli formula (see for instance [Mil94] or [Vin93]) relates the variation of V_t and of the angles $\theta_{i,t}$ as follows:

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Schläfli's formula:

$$\sum_i W_{i,t}\frac{d\theta_{i,t}}{dt} = mK\frac{dV_t}{dt} \tag{1}$$

This formula has been extended and used on several occasions recently, see for instance [CR96], [SP97], [Bon]. The first goal of this paper is to extend (1) to deformations of smooth hypersurfaces in H^n. A remarkable point is that the formula which appears remains valid for deformations of hypersurfaces in Einstein manifolds. In section 2, we extend it also to a much more general situation, namely the deformations of the Einstein metric in an Einstein manifold with boundary.

When the $W_{i,t}$ are constant, the left-hand side of (1) is a polyhedral analogue of the variation of the mean curvature integral of a hypersurface. Indeed, the polyhedral analogue of the mean curvature integral is $H = \sum W_i\theta_i$ (where θ_i is the *exterior* dihedral angle at the i-th codimension-2 face). Using the product rule, we see that

$$\frac{dH}{dt} = \sum \frac{dW_i}{dt}\theta_i + \sum W_i\frac{d\theta_i}{dt}.$$

When the deformation is isometric, the first sum on the right hand side vanishes, and, combining that with the formula 1, we see that

$$\frac{dH}{dt} = -mK\frac{dV_t}{dt},$$

where the minus sign comes from replacing the dihedral angles with exterior dihedral angles.

When $K = 0$, the right-hand side is 0. This shows that the "mean curvature" of a 1-parameter family of Euclidean polyhedra with constant induced metric is constant. This has been used in [AR97] to prove, using geometric measure theory methods, a similar result for isometric deformations of smooth hypersurfaces in space-forms; we prove this result directly by differential-geometric methods here (in the more general setting of Einstein manifolds), indeed, we show the rather strong Theorem 7. Theorem 5 should be compared with the celebrated mean curvature variation (under bending) formula of Herglotz; see [BG93].

Isometric deformations of (smooth) hypersurfaces remain rather mysterious. It is known since Liebmann [Lie00] (also see [Her79], [Spi75, vol V]) that (strictly) convex surfaces in $\mathbf{R}^3$ admit no such deformation, and this has been extended to S^3 and H^3 by Pogorelov [Pog73]. On the other hand, it is unknown whether any smooth closed surface in $\mathbf{R}^3, S^3$ or H^3 has a 1-parameter family of isometric deformations, or even if any closed surface with no open flat region has a smooth infinitesimal isometric deformation. But it *is* known that there exist non-rigid polyhedra, see [Con77] and [Ble96].

A smooth version of (1) leads to some rigidity results. Aside from the remarks following Theorem 5, we prove in Section 3 a rigidity result for Ricci-flat manifolds with umbilic boundaries, with respect to Einstein deformations of the metric which leave the induced metric on the boundary fixed.

Section 4 contains some applications of our "Schläfli formula" to codimension one foliations of Einstein manifolds. Section 5 recalls some concepts of integral geometry, to see how some of our results may be recast in integral-geometric terms. In This is particularly relevant to section 6 (after giving brief preliminaries in Section 5), where we extend a classical inequality of Alexandrov for convex bodies in Euclidean space to the hyperbolic and spherical settings.

This paper can be considered (among other things) as a hint that some important elements of hyperbolic geometry in dimension 3 can be extended in higher dimension in the setting of Einstein manifolds (with negative curvature). Another such hint is given in [Sch98], which contains a partial extension of some classical results of the theory of convex surfaces in hyperbolic 3-space to Einstein manifolds with boundary.

The smooth Schläfli formula given in section 1 also has analogous "higher" smooth Schläfli formulas (see [SS99]) but in constant curvature manifolds only. Going to the polyhedral case leads

to "higher" polyhedral Schläfli formulas, relating the variations of the volumes of the p-faces to the variations of the "curvatures" of the $(p+2)$-faces. Similar (polyhedral) formulas were given (in some special cases) in [SP97].

Throughout this paper, M is an Einstein manifold of dimension $m+1 \geq 3$, and D is its Levi-Civita connection. When dealing with a hypersurface Σ (resp. with the boundary ∂M), we call I the *induced metric*, also called the first fundamental form, of the corresponding immersion in M. $\overline{D}$ is the Levi-Civita connection of I, and B the shape operator defined, for any $x \in \Sigma$ (resp. $x \in \partial M$) and $X \in T_x\Sigma$ (resp. $X \in T_x\partial M$) by

$$BX = -D_X n \ ,$$

where n is the oriented normal unit vector to Σ (resp. the unit exterior normal to ∂M). The second fundamental form $I\!I$ of Σ (resp. ∂M) is defined by

$$I\!I(X,Y) = I(X, BY) \ ,$$

and the third fundamental form by

$$I\!I\!I(X,Y) = I(BX, BY) \ .$$

The trace H of B is called the "mean curvature" (some definitions differ by a factor m) and the "higher mean curvatures" $H_k, k \geq 1$, are the higher symmetric functions of the principal curvatures of Σ (resp. ∂M). For instance, $H_2 = (H^2 - \text{tr}(B^2))/2$. dV, dA are the volume elements in M and on Σ (resp. ∂M) respectively.

We denote the divergence acting on symmetric tensors by δ, and its formal adjoint by δ^*. Therefore, if h is a symmetric 2-tensor and $(e_i)_{i \in \mathbf{N}_{m+1}}$ an orthonormal moving frame on M, then, for any vector $x \in TM$:

$$(\delta h)(x) = -\sum_i (D_{e_i}h)(e_i, x) \ ,$$

and, if v is a vector field, $p \in M, x, y \in T_pM$, then:

$$(\delta^* v)(x,y) = \frac{1}{2}(\langle D_x v, y\rangle + \langle D_y v, x\rangle) \ .$$

We will often implicitly identify (through the metric) vector fields and 1-forms, as well as quadratic forms and linear morphisms.

1 Deformation of hypersurfaces

This section contains an analogue of the Schläfli formula for deformations of (smooth) hypersurfaces in a *fixed* Einstein manifold M, which can be Riemannian or Lorentzian (the other pseudo-Riemannian cases can be treated in the same way; we have not included them to keep things as simple as possible). This contrasts with the results in the next section, where the same formula is proved for variations of the metric inside a manifold with boundary (which is much more general) but only when M is Riemannian.

We also show how this "smooth" Schläfli formula can be used to recover the classical polyhedral formula (1), in the Riemannian and Lorentzian cases.

The techniques here are quite elementary, and use the method of moving frames.

Theorem 1 *Let Σ be a smooth oriented hypersurface in a (Riemannian) Einstein $(m+1)$-manifold M with scalar curvature S, and v a section of the restriction of TM to Σ. v defines a deformation of Σ in M, which induces variations V', H' and I' of the volume bounded by Σ, mean curvature, and induced metric on Σ. Then:*

$$\frac{S}{m+1}V' = \int_\Sigma H' + \frac{1}{2}\langle I', I\!I\rangle dA \tag{2}$$

Σ actually does not need bound a finite volume domain for this formula to hold. If it doesn't, then V doesn't exist, but its variation still makes sense (since Σ is homologous to its images under deformations).

Proof: We first prove the formula for v tangent to Σ, then we'll check for normal vector fields. When v is tangent to Σ, $V' = 0$, and

$$I'(X,Y) = \langle D_X v, Y\rangle + \langle D_Y v, X\rangle = 2(\delta^* v)(X,Y)$$

so that:

$$\int_\Sigma \langle I', I\!I\rangle dA = 2\int_\Sigma \langle \delta^* v, I\!I\rangle dA = 2\int_\Sigma \langle v, \delta I\!I\rangle dA$$

Let (e_i) be an orthonormal frame for I for which B is diagonal. The Codazzi equation shows that

$$(\overline{D}_X I\!I)(Y,Z) = (\overline{D}_Y I\!I)(X,Z) + \langle R(X,Y)n, Z\rangle$$

so

$$
\begin{aligned}
\langle \delta I\!I, v\rangle &= -(\overline{D}_{e_i} I\!I)(e_i, v)\\
&= -(\overline{D}_v I\!I)(e_i, e_i) - \langle R(e_i, v)n, e_i\rangle\\
&= -dH(v) + \mathrm{ric}(v, n)
\end{aligned}
$$

Now M is Einstein and n is orthogonal to v, so that:

$$\langle \delta I\!I, v\rangle = -dH(v) \tag{3}$$

Therefore:

$$\int_\Sigma \langle I', I\!I\rangle dA = -2\int_\Sigma dH(v)dA$$

This proves the formula when v is tangent to Σ.

Suppose now that v is a normal vector field, i.e. $v = fn$ for some function f on Σ. Since f is the difference between two strictly positive functions, it is enough to prove the result when f does not vanish. Let x, y be vector fields on Σ. Choose an extension of fn to some vector field on a neighborhood Ω of Σ in M, with n the unit orthogonal to the image of Σ by the flow of fn, and $df(n) = 0$. Extend x, y to Ω by the flow of fn, then $[fn, x] = [fn, y] = 0$. We now have:

$$I'(x,y) = fn.\langle x, y\rangle = \langle D_{fn} x, y\rangle + \langle x, D_{fn} y\rangle = \langle D_x(fn), y\rangle + \langle x, D_y(fn)\rangle = --- -2f I\!I(x,y)$$

so $I' = -2f I\!I$. One also checks that $D_{fn} n = -Df$, so that:

$$
\begin{aligned}
I\!I'(x,y) &= -fn.\langle D_x n, y\rangle\\
&= -\langle D_{fn} D_x n, y\rangle - \langle D_x n, D_{fn} y\rangle\\
&= -\langle D_x D_{fn} n + R_{fn,x} n + D_{[fn,x]} n, y\rangle - \langle D_x n, D_y(fn)\rangle\\
&= \langle D_x Df, y\rangle - \langle R_{fn,x} n, y\rangle - f I\!I\!I(x,y)
\end{aligned}
$$

and

$$I\!I' = H_f - f\langle R_{n,.} n, \cdot\rangle - f I\!I\!I \tag{4}$$

where H_f is the Hessian of f on Σ.

Taking the trace of this equation:

$$H' = \mathrm{tr}(I\!I') - \langle I', I\!I\rangle = -\Delta f + f\mathrm{ric}(n,n) - f\mathrm{tr}(I\!I\!I) - \langle I', I\!I\rangle$$

Now the integral over Σ of Δf is zero, and the integral of f is V' because the deformation is normal. The result follows, because $I' = -2f I\!I$, so that $-2f\mathrm{tr}(I\!I\!I) = \langle I', I\!I\rangle$. $\square$

This formula leads easily to the "classical" Schläfli formula for polyhedra in space-forms:

Theorem 2 *Let P be a convex polyhedron in a $(m+1)$-dimensional space-form M with scalar curvature S; for any deformation of P, the variation V' of the volume bounded by P is given in term of the variations θ'_i of the dihedral angles at the codimension 2 faces by:*

$$\frac{S}{m+1}V' = \sum_i W_i\theta'_i$$

where W_i is the $(m-1)$-volume of the codimension 2 face i.

Proof: First note that Theorem 1 also applies for deformations of a $C^{1,1}$, piecewise smooth hypersurface (if the deformation preserves the decomposition into smooth parts). This is proved by an easy approximation argument. The formula remains the same, and each term make sense in this case.

Call P_ϵ the set of points at distance ϵ of P on the outside (i.e. on the side of P which is concave). For ϵ small enough, P_ϵ is a $C^{1,1}$, piecewise smooth hypersurface, and we can apply Theorem 1. Note $I'_\epsilon, II_\epsilon, H'_\epsilon, V'_\epsilon$ the quantities corresponding to I', II, H', V' for P_ϵ. Then:

$$\frac{S}{m+1}V'_\epsilon = \int_{P_\epsilon} H' + \frac{1}{2}\langle I'_\epsilon, II_\epsilon \rangle dA .$$

For ϵ small enough, we can decompose P_ϵ as

$$P_\epsilon = \cup_{k=1}^{m+1} P_{\epsilon,k} ,$$

where $P_{\epsilon,k}$ is the set of points where the normal meets P on a codimension k face. Using the flow of the unit normal vectors to the P_ϵ, we can also identify P_ϵ and $P_{\epsilon'}$ for $\epsilon' \neq \epsilon$, so that we can consider e.g. I'_ϵ as a 1-parameter family of symmetric 2-tensors on a fixed manifold.

If $x \in P_{\epsilon,2}$, then the normal to P_ϵ at x meets some codimension 2 face F_i of P; let $\alpha_{i,t}$ be the dihedral angle at F_i. If $v, w \in T_x P_\epsilon$ correspond to vectors orthogonal to TF, then

$$I'_\epsilon(v,w) \simeq \frac{2}{\alpha_{i,t}}\frac{d\alpha_{i,t}}{dt}I_\epsilon(v,w)$$

as $\epsilon \to 0$. On the other hand,

$$II_\epsilon(v,w) \simeq \frac{1}{\epsilon}I_\epsilon(v,w)$$

If v, w now correspond to vectors in TF_i, then

$$I'_\epsilon(v,w) = O(1) ,$$

while

$$II_\epsilon(v,w) = O(\epsilon) .$$

Using those 2 cases, we see that, at any point in $P_{\epsilon,2}$:

$$\langle I'_\epsilon, II_\epsilon \rangle \simeq \frac{2}{\epsilon\alpha_{i,t}}\frac{d\alpha_{i,t}}{dt} .$$

Now the volume element of $P_{\epsilon,2}$ is equivalent to ϵ as $\epsilon \to 0$, so:

$$\lim_{\epsilon \to 0} \int_{P_{\epsilon,2}} \langle I'_\epsilon, II_\epsilon \rangle dA = \sum_i 2W_{i,t}\frac{d\alpha_{i,t}}{dt} .$$

For $P_{\epsilon,1}$ (that is, for codimension 1 faces), only vectors parallel to the faces have to be taken into account, and their contribution is of order $O(\epsilon)$ (as above for $P_{\epsilon,2}$).

For $P_{\epsilon,k}$ with $k \geq 3$, the same reasoning shows that only vectors orthogonal to the faces count; if v, w are such vectors, then

$$I'_\epsilon(v, w) = O(I_\epsilon(v, w)) \ ,$$

while

$$\mathit{II}_\epsilon(v, w) = O(I_\epsilon(v, w)/\epsilon) \ ,$$

and the volume element on $P_{\epsilon,k}$ is as $O(\epsilon^{k-1})$, so

$$\lim_{\epsilon \to 0} \int_{P_{\epsilon,k}} \langle I'_\epsilon, \mathit{II}_\epsilon \rangle dA = 0 \ .$$

It is also easy to check that

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} H'_\epsilon dA = 0 \ ,$$

and this leads to the formula. $\square$

Of course, P does not need to be convex: once the corollary is proved for convex polyhedra, it is clear that it also applies to non-convex ones, since they can be decomposed into convex pieces.

The proof of Theorem 1 also applies to the Lorentzian case. The only difference is that now $g(n, n) = -1$, so the volume variation has a minus sign in the formula.

Theorem 3 *Let Σ be a smooth oriented space-like hypersurface in a Lorentzian Einstein $(m+1)$-manifold (M, g) with $ric_g = mkg$, and let v be a section of the restriction of TM to Σ. v defines a deformation of Σ in M, which induces variations V', H' and I' of the volume bounded by Σ, mean curvature, and induced metric on Σ. Then:*

$$--- mkV' = \int_\Sigma H' + \frac{1}{2}\langle I', \mathit{II} \rangle dA \tag{5}$$

Here again, the volume might be defined only up to an additive constant (for instance as the volume bounded by Σ and some fixed homologous hypersurface Σ_0), but its variation is well defined. For instance, if Σ is a compact space-like hypersurface in the de Sitter space, its "volume" can be defined as the oriented volume of the domain bounded by Σ and by some space-like totally geodesic hyperplane S_0. This volume actually does not depend on S_0, because if S_1 is some other totally geodesic hyperplane, then, as (5) shows, the oriented volume of the domain bounded by S_0 and S_1 is zero.

This lemma could actually be extended almost without change to other pseudo-Riemannian manifolds, and also to hypersurfaces which are not space-like.

Applying this lemma to the set of points at distance ϵ from a polyhedron in S_1^n (as above in Theorem 2), one obtains the Schläfli formula for de Sitter polyhedra as in [SP97] (where it was proved for simplices using a more combinatorial approach).

Theorem 4 *Let P be a convex space-like polyhedron in the de Sitter space S_1^{m+1}, which is dual to a hyperbolic polyhedron. For any deformation of P, the variation V' of the volume bounded by P is given in term of the variations θ'_i of the dihedral angles at the codimension 2 faces by:*

$$mV' + \sum_i W_i \theta'_i = 0$$

where W_i is the $(m-1)$-volume of the codimension 2 face i.

The conditions that P is convex and dual to a hyperbolic polyhedron are actually not necessary, and the formula even remains valid for many polyhedra that are not space-like. It then helps to use a definition of angles and volume well adapted to this Lorentzian setting, i.e. with complex

values (as in [Sch]). It is not obvious how to give a complete proof using smooth formulas (as above) but many cases can be treated simply by using sums or differences of polyhedra for which smooth formulas work. For instance, this can be done for all space-like polyhedra.

Theorem 1 applied in Euclidean space leads to the

Theorem 5 *In $\mathbf{R}^{m+1}$, the integral of the mean curvature remains constant under an isometric deformation of a compact hypersurface.*

On the other hand, the integral mean curvature is **not** determined by the metric on ∂M: this is already visible in $\mathbf{R}^3$. Namely, some metrics on S^2 admit two isometric embeddings in $\mathbf{R}^3$: the classical example is that a (topological) sphere in $\mathbf{R}^3$ which is tangent to a plane along a circle can be "flipped" so as to obtain another embedding with the same induced metric [Spi75]. Those two embeddings do not in general have the same integral mean curvature – and thus we have a complicated way of seeing that the two flipped surfaces cannot be bent one into the other.

The analogue of Theorem 5 is also true, but in a pointwise sense, for the higher mean curvatures:

Theorem 6 *In $\mathbf{R}^{m+1}$, the integral of H_k $(k \geq 2)$ remains constant in an isometric deformation of a hypersurface.*

This comes from the following (probably classical) description of the possible isometric deformations of a hypersurface for $m + 1 \geq 4$:

Remark 1 *Let $(\Sigma_t)_{t \in [0,1]}$ be a 1-parameter family of hypersurfaces in a space-form, such that the induced metric I_t is constant to the first order at $t = 0$. Then, at each point, one of the following is true:*

- *$I\!I_0 = 0$;*

- *$rk(I\!I_0) \leq 2$, and $I\!I_0'$ vanishes on the kernel of $I\!I_0$;*

- *$I\!I_0' = 0$.*

where $I\!I_t$ is the second fundamental form of Σ_t, and $I\!I_t'$ its variation.

Theorem 6 clearly follows, because H_k' is zero for $k \geq 3$ in each case, and the Gauss formula gives the proof for $k = 2$.

Proof of Remark 1: Choose an orthonormal frame $(e_1, \cdots, e_m)$ on Σ_0 for which $I\!I_0$ is diagonal, with eigenvalues $(k_1, \cdots, k_m)$. By the Gauss formula, $I\!I_t \oslash I\!I_t$ (where $\oslash$ is the Kulkarni-Nomizu product) is determined by the induced metric, and is thus independent on t. Therefore, for any choice of indices p, q, r, s:

$$I\!I_0(e_p, e_s)I\!I_0'(e_q, e_r) + I\!I_0'(e_p, e_s)I\!I_0(e_q, e_r) = I\!I_0(e_p, e_r)I\!I_0'(e_q, e_s) + I\!I_0'(e_p, e_r)I\!I_0(e_q, e_s)$$

Taking p, q, r distinct but $s = p$ shows that

$$k_p I\!I_0'(e_q, e_r) = 0 \tag{6}$$

while taking $p = s \neq q = r$ leads to:

$$k_p I\!I_0'(e_q, e_q) + k_q I\!I_0'(e_p, e_p) = 0 \tag{7}$$

Consider the case where $\mathrm{rk}(I\!I_0) \geq 3$. For each choice of p, q, r with $k_p, k_q, k_r \neq 0$, adding eq. (7) (divided by $k_p k_q$) for the pairs (p, q) and (p, r) and subtracting the same equation for the pair (q, r) shows that $I\!I_0'(e_p, e_p) = 0$, and the same for q, r, so we already see that all diagonal terms of $I\!I_0'$ are zero. Then eq. (6) shows that all non-diagonal terms are zero too, so $I\!I_0' = 0$.

If $\text{rk}(I\!I_0) \leq 2$ but $I\!I_0 \neq 0$, then eq. (6) and eq. (7) easily show that $I\!I_0' = 0$ except maybe in the subspace generated by the eigenvectors of $I\!I_0$ with non-zero eigenvalue. $\square$

Theorems 5 and 6 can be combined to give the following geometric statement. Denote by Σ_t^ϵ the parallel surface at distance ϵ from Σ_t. It is well-known (see, eg, Santalo's book [San76]) that the area of Σ^ϵ is a polynomial in ϵ where the coefficient of ϵ^k is (essentially) the k-th mean curvature of Σ. The two Theorems 5 and 6 can than be combined as stating that:

Theorem 7 *The area of Σ_t^ϵ stays constant when Σ_t is a bending of Σ_0.*

2   Einstein manifolds with boundary

In this section, $(M, \partial M)$ is a compact manifold with boundary with an Einstein metric g of scalar curvature S. We will prove the same formula as in the previous section, but in a much more general setting: instead of moving a hypersurface in an Einstein manifold, we will be changing the metric (among Einstein metrics of given scalar curvature) inside this manifold with boundary. Although the two operations are equivalent in dimension at most 3, moving the inside metric is much more general in higher dimension. On the other hand, our proof only works for Riemannian Einstein manifolds. It is not obvious whether it can be extended to the pseudo-riemannian setting.

As always when studying deformations of Riemannian metrics, we need put some kind of restriction to remove the indeterminacy coming from the fact that some deformations are geometrically trivial, that is, they just correspond to the action of vector fields on the metric. We prevent those deformations in the same way as e.g. in [GL91], [DeT81] or [Biq97], that is, we only consider metric variations h such that $2\delta h + d\text{tr} h = 0$. The following proposition shows that we don't forget any metric variation when doing this.

Proposition 1 *Let h' be a smooth variation of g. Suppose that either $S \leq 0$, or that M is strictly convex. There exists another smooth variation h of g such that $2\delta h + d\text{tr}(h) = 0$ and that $h = h' + \delta^* v_0$, where v_0 is a vector field vanishing on ∂M.*

Proof: Suppose v is a vector field on M, let $h = h' + \delta^* v$. Then

$$2\delta h + d\text{tr} h = 2\delta h' + d\text{tr} h' + 2\delta(\delta^* v) + d\text{tr}(\delta^* v) \ .$$

Now, if x is a vector field on M:

$$
\begin{aligned}
2\delta(\delta^* v)(x) &= -\sum_i 2(D_{e_i}(\delta^* v))(e_i, x) \\
&= \sum_i -e_i.(2\delta^* v)(e_i, x) + (2\delta^* v)(D_{e_i} e_i, x) + (2\delta^* v)(e_i, D_{e_i} x) \ ,
\end{aligned}
$$

so

$$2\delta(\delta^* v)(x) = \sum_i -e_i.(\langle D_{e_i} v, x \rangle + \langle D_x v, e_i \rangle) +$$

$$+ \langle D_{D_{e_i} e_i} v, x \rangle + \langle D_x v, D_{e_i} e_i \rangle + \langle D_{e_i} v, D_{e_i} x \rangle + \langle D_{D_{e_i} x} v, e_i \rangle$$

and

$$2\delta(\delta^* v)(x) = \sum_i \langle -D_{e_i} D_{e_i} v + D_{D_{e_i} e_i} v, x \rangle + \langle -D_{e_i} D_x v + D_{D_{e_i} x} v, e_i \rangle \ .$$

On the other hand:

$$d(\text{tr}(\delta^* v))(x) = x.\left(\sum_i \langle D_{e_i} v, e_i \rangle \right) = \sum_i \langle D_x D_{e_i} v, e_i \rangle + \langle D_{e_i} v, D_x e_i \rangle \ ,$$

so that

$$(2\delta(\delta^*v)+d\mathrm{tr}(\delta^*v))(x) = \sum_i \langle ---D_{e_i}D_{e_i}v+D_{D_{e_i}e_i}v, x\rangle - \langle R_{e_i,x}v, e_i\rangle + \langle D_{D_x e_i}v, e_i\rangle + \langle D_{e_i}v, D_x e_i\rangle .$$

If ω is the connection form of the frame $(e_i)_{i\in \mathbf{N}_{m+1}}$, then

$$\sum_i \langle D_{D_x e_i}v, e_i\rangle + \langle D_{e_i}v, D_x e_i\rangle = \sum_i (2\delta^*v)(e_i, D_x e_i) = \langle 2\delta^*v, \omega(x)\rangle = 0$$

because δ^*v is symmetric and $\omega(x)$ is skew-symmetric. Therefore,

$$(2\delta(\delta^*v) + d\mathrm{tr}(\delta^*v))(x) = \langle D^*Dv, x\rangle - \mathrm{ric}(v, x) = \langle D^*Dv, x\rangle - \frac{S}{m+1}\langle v, x\rangle ,$$

so

$$2\delta(\delta^*v) + d\mathrm{tr}(\delta^*v) = D^*Dv - \frac{S}{m+1}v . \tag{8}$$

To prove the proposition, we have to solve the elliptic problem:

$$\begin{cases} D^*Dv - \frac{S}{m+1}v &= -(2\delta h' + d\mathrm{tr}(h')) \\ v_{|\partial M} &= 0 \end{cases} \tag{9}$$

Call $\Gamma_0^1 TM$ the space of vector fields on M which are in the Sobolev space H^1 and whose trace on ∂M vanishes (this essentially means that they are zero on ∂M), and define

$$\begin{aligned} F: \quad \Gamma_0^1 TM &\rightarrow \mathbf{R} \\ v &\mapsto \tfrac{1}{2}\int_M \langle Dv, Dv\rangle - \tfrac{S}{m+1}\langle v, v\rangle dV + \int_M \langle 2\delta h' + d\mathrm{tr}(h'), v\rangle dV . \end{aligned}$$

Then F is strictly convex, and moreover it is coercive; this is clear if $S < 0$, and, if $S = 0$, it follows from the Poincaré inequality for vector fields vanishing on ∂M:

$$\exists C, \forall v \in \Gamma_0^1 TM, \int_M \langle Dv, Dv\rangle dV \geq C \int_M \langle v, v\rangle dV$$

If $S > 0$, a more careful argument is necessary. Let $u := \|v\|$. Then

$$\langle Du, Du\rangle \leq \langle Dv, Dv\rangle$$

and

$$\int_M \langle Du, Du\rangle dV \geq \lambda_1 \int_M u^2 dV ,$$

where λ_1 is the first eigenvalue of the Dirichlet problem for the Laplacian on M. But it is known (see [Rei77], [Kas84]) that, for M convex, and under the hypothesis that the Ricci curvature is bounded below by $S/(m+1)$

$$\lambda_1 \geq \frac{S}{m} ,$$

with equality if and only if M is a hemisphere. Therefore,

$$\int_M \langle Dv, Dv\rangle dV \geq \frac{S}{m}\int_M \langle v, v\rangle dV$$

and F is again coercive.

Therefore, F admits a unique minimum v_0 on $\Gamma_0^1 TM$, which is smooth by standard elliptic arguments. Then, for all $u \in \Gamma TM$,

$$(T_{v_0} F)(u) = 0$$

so that

$$\int_M \langle Dv_0, Du \rangle - \frac{S}{m+1} \langle v_0, u \rangle dV + \int_M \langle 2\delta h' + dtr(h'), u \rangle dV =$$

$$= \int_M \langle D^* Dv_0 - \frac{S}{m+1} v_0 + 2\delta h' + dtr(h'), u \rangle dV = 0$$

and $D^* Dv_0 - \frac{S}{m+1} v_0 = -2\delta h' - dtr(h')$ as needed. $\square$

Another way of solving eq. (9) would be to check that it has index 0, and that if $2\delta(\delta^* v) + dtr(\delta^* v) = 0$ on M and $v = 0$ on ∂M, then $v \equiv 0$.

If g is an Einstein metric, we say that a 2-tensor h is an "Einstein variation" of g if the associated variation of the metric induces a variation of the Ricci tensor which is proportional to h, so that $g + \epsilon h$ remains, to the first order, an Einstein manifold with constant scalar curvature.

Theorem 8 *Let h be a smooth Einstein variation of g. Then:*

$$\frac{S}{m+1} V' = \int_{\partial M} H' + \frac{1}{2} \langle h_{|\partial M}, I\!I \rangle dA \tag{10}$$

Proof: By the previous proposition, we can suppose that $2\delta h + dtr(h) = 0$. First, we compute the variation $I\!I'$ of $I\!I$ on ∂M. Let x be a vector field on M so that $D_n x = 0$. Extend n to a unit vector field on M such that $D_n n = 0$. Then

$$2I\!I(x, x) = -2\langle D_x n, x \rangle = -n.\langle x, x \rangle - - - 2\langle [x, n], x \rangle .$$

Now, since n remains the unit normal to ∂M

$$n' = -\frac{\tau}{2} n - a$$

where $\tau = h(n, n)$ and $a \in T\partial M$ is such that for any vector $y \in T\partial M$, $\langle y, a \rangle = h(n, y)$. Therefore,

$$
\begin{aligned}
2I\!I'(x, x) &= -n.h(x, x) - 2h([x, n], x) + \left(\frac{\tau}{2} n + a \right).\langle x, x \rangle + \langle [x, \tau n + 2a], x \rangle \\
&= -(D_n h)(x, x) + 2h(Bx, x) + a.\langle x, x \rangle - \tau \langle Bx, x \rangle + 2\langle [x, a], x \rangle \\
&= -(D_n h)(x, x) + 2h(Bx, x) + 2\langle D_x a, x \rangle - \tau \langle Bx, x \rangle .
\end{aligned}
$$

To go further, we note $\overline{\delta}$ the divergence on ∂M, α the 1-form dual to a on ∂M, and $t := tr(h)$.

If $(u_1, \cdots, u_n)$ is an orthonormal frame on ∂M for which $I\!I$ is diagonal, extended on M so that $D_n u_i = 0$, we have:

$$2tr(I\!I') = -\sum_i (D_n h)(u_i, u_i) + 2\langle h, I\!I \rangle - - - 2(\overline{\delta}\alpha) - \tau tr(I\!I) .$$

But

$$
\begin{aligned}
- - - \sum_i (D_n h)(u_i, u_i) &= -dt(n) + (D_n h)(n, n) \\
&= -dt(n) - (\delta h)(n) - \sum_i (D_{u_i} h)(u_i, n) \\
&= -\frac{dt(n)}{2} - \sum_i (D_{u_i} h)(u_i, n) \\
&= -\frac{dt(n)}{2} - \sum_i u_i.\alpha(u_i) + h(D_{u_i} u_i, n) + h(u_i, D_{u_i} n) \\
&= -\frac{dt(n)}{2} - \sum_i (\overline{D}_{u_i} \alpha)(u_i) + I\!I(u_i, u_i)\tau - h(u_i, Bu_i)
\end{aligned}
$$

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and, finally,

$$2\mathrm{tr}(I\!I') = -\frac{dt(n)}{2} + \langle h, I\!I \rangle - (\overline{\delta}\alpha) \ . \tag{11}$$

The statement that h is an Einstein variation of g can be written (see [Bes87], chapter 12) as the equation:

$$D^*Dh - 2\overline{R}h - 2\delta^*\delta h - Ddt = 0 \ ,$$

where $\overline{R}$ is the curvature operator acting on symmetric 2-tensors; and, since $2\delta h + dt = 0$:

$$D^*Dh - 2\overline{R}h = 0 \ .$$

Taking the trace of this equation, we find that:

$$\Delta t = \frac{2S}{m+1}t \ .$$

An elementary computation shows that the variation of the volume of M is equal to half the integral of the trace t of h:

$$2V' = \int_M tdV \ .$$

But

$$\frac{2S}{m+1}\int_M tdV = \int_M \Delta tdV = -\int_{\partial M} dt(n)dA$$

and, using eq. (11), we obtain:

$$2\int_{\partial M} H'dA = \int_{\partial M} 2\mathrm{tr}(I\!I') - 2\langle h_{|\partial M}, I\!I \rangle dA =$$

$$= -\int_{\partial M} \frac{dt(n)}{2} + \langle h_{|\partial M}, I\!I \rangle + (\overline{\delta}\alpha)dA = \frac{2S}{m+1}V' - \int_{\partial M} \langle h_{|\partial M}, I\!I \rangle dA$$

from which the result follows. $\square$

Formula (10) is even simpler for variations which vanish on ∂M:

Theorem 9 *If h is a smooth Einstein variation of g which does not change the induced metric on ∂M, then*

$$\int_{\partial M} H'dV = \frac{S}{m+1}V'$$

In particular, for $S = 0$, this implies that the integral of the mean curvature of the boundary is constant under an Einstein variation which does not change the induced metric on ∂M; this is a direct generalization of Theorem 5.

A more interesting application can be found by looking at "singular objects", just as we did to get the polyhedral Theorem 2 from the smooth Lemma 1. There are no polyhedra in general Einstein manifolds, but we can check what happens when we deform Einstein manifolds with cone singularities. It should be pointed out that some of the most interesting modern uses of the classical Schläfli formula concern hyperbolic 3-dimensional cone-manifolds.

Let M be a compact $(m+1)$-manifold, and N a compact codimension 2 submanifold of M. Suppose (g_t) is a 1-parameter family of Einstein metrics with fixed scalar curvature $S \le 0$ on $M \setminus N$, with a conical singularity on N in the sense that, in normal coordinates around N, g_t has an expansion like:

$$g_t = h_t + dr^2 + r^2 d\theta^2 + o(r^2)$$

where h_t is the metric induced on N by g_t, and $\theta \in \mathbf{R}/\alpha_t \mathbf{Z}$ for some $\alpha_t \in \mathbf{R}$. Call V_t the volume of $(M \setminus N, g_t)$, and W_t the volume of (N, h_t). Then:

Corollary 1 V_t *varies as follows:*

$$\frac{S}{m+1}\frac{dV_t}{dt} = W_t\frac{d\alpha_t}{dt} \ .$$

The same formula of course remains true if N has several connected components, each with a different value of α_t.

Proof: It is similar to the proof of Theorem 2, so we go a little faster. Set

$$N_\epsilon(t) = \{x \in M \ , \ d(x,N) \geq \epsilon\alpha_t\}$$

Apply Theorem 8 to the boundary of $N_\epsilon(t)$ and take the limit as $\epsilon \to 0$. Again, we consider ∂N_ϵ as a fixed manifold (diffeomorphic to $N \times S^1$) with a 1-parameter family of metrics depending on ϵ. If $v, w \in T\partial N_\epsilon$ correspond to vectors in TN, then

$$I'_\epsilon(v,w) = O\left(1\right)$$

while

$$I\!I_\epsilon(v,w) = O(\epsilon)$$

If $v \in T\partial N_\epsilon$ corresponds to a vector normal to N, then

$$I'_\epsilon(v,v) \simeq \frac{2}{\alpha_t}\frac{d\alpha_t}{dt}I_\epsilon(v,v)$$

while

$$I\!I_\epsilon(v,v) \simeq \frac{I_\epsilon(v,v)}{\epsilon}$$

so that finally

$$\langle I'_\epsilon, I\!I_\epsilon \rangle \simeq \frac{2}{\epsilon\alpha_t}\frac{d\alpha_t}{dt}$$

On the other hand, the mean curvature of the boundary (given only by the $\partial/\partial\theta$ direction) is

$$H_\epsilon = \frac{1}{\epsilon} + o\left(\frac{1}{\epsilon}\right)$$

and

$$H'_\epsilon = o\left(\frac{1}{\epsilon}\right)$$

so that we finally find that

$$\frac{S}{m+1}V'_t = \lim_{\epsilon \to 0}\int_{\partial N_\epsilon(t)}\frac{1}{2}\langle I'_\epsilon, I\!I_\epsilon \rangle dA = \frac{d\alpha_t}{dt}W_t$$

□

Note: The same computation could be made with N replaced by a stratified submanifold; the same result follows.

Example: take $m + 1 = 3$ in the previous example. We find the Schläfli formula for the variation of the volume of a hyperbolic cone-manifold.

3 Applications to rigidity

In this section, we use the Schläfli formula above to prove a rigidity result for Ricci-flat manifolds with umbilic boundary; it is a generalization of the classical result (see [Spi75]) that the round sphere is rigid in $\mathbf{R}^3$, that is, it can not be deformed smoothly without changing its induced metric.

This kind of rigidity result could be used in the future to prove that, given a Ricci-flat manifold M with umbilic boundary and induced metric g_0 on the boundary, any metric close to g_0 on ∂M can be realized as induced on ∂M by some Ricci-flat metric on M. In this setting, rigidity corresponds to the local injectivity of an operator sending the metrics on M to the metrics on ∂M. In dimension 3, this would be a part of the classical result (see [Nir53]) that metrics with curvature $K > 0$ on S^2 can be realized as induced by immersions into $\mathbf{R}^3$. This circle of ideas is illustrated in [Sch98]. It is rather remarkable that the same condition (that the boundary is umbilic) appears both here and in [Sch98], in the same kind of rigidity questions, but in a very different way.

The first point is to understand what an umbilic hypersurface in an Einstein manifold is. By definition, if N is a Riemannian manifold and is S a hypersurface, then S is umbilic if, at each point $s \in S$, $I\!I$ is proportional to I, with a proportionality constant $\lambda(s)$ depending on s. Now

Remark 2 *If N is Einstein, then λ is constant on each connected component of S.*

Proof: Let B be the shape operator of S, and n the unit normal. For $s \in S$ and $x, y \in T_s S$, the Codazzi formula asserts that:
$$(d^{\overline{D}} B)(x, y) = R_{x,y} n$$
where $\overline{D}$ is the Levi-Civita connection of S, and R the curvature operator of N. Since $I\!I = \lambda I$, this means that:
$$(d\lambda(x))y - (d\lambda(y))x = R_{x,y} n$$
Let $(e_i)_{1 \le i \le m}$ be a moving frame on S. Taking the trace of the previous expression with respect to y shows that:
$$m d\lambda(x) - \sum_{i=1}^{m} d\lambda(e_i)\langle x, e_i \rangle = -{-}{-}\mathrm{ric}(x, n)$$

and $\mathrm{ric}(x, n) = 0$ because N is Einstein and n is orthogonal to x. Therefore:

$$(m-1)d\lambda(x) = 0$$

for any tangent vector s to S, so λ is locally constant. $\square$

Corollary 2 *Umbilic hypersurfaces of Einstein manifolds are analytic.*

A rather clumsy way to prove this is to note that, because of the previous remark, umbilic hypersurfaces are locally graphs of solutions of some elliptic PDE with analytic coefficients (because Einstein metrics are analytic, see [Bes87]). A classical elliptic smoothness theorem then gives the result. As a consequence:

Corollary 3 *Let (Σ, h) be a compact analytic Riemannian m-manifold, and $\lambda, S \in \mathbf{R}$. There exists at most one germ of Einstein $(m+1)$-manifold around Σ with scalar curvature S which induces h and for which Σ is umbilic with $I\!I = \lambda I$.*

Proof: Let M be such a germ of Einstein manifold around Σ, and g its metric. By taking the geodesic flow of the exponential normal to Σ in M, we see that g can be locally written, in some neighborhood V of Σ, as
$$g = k_t + dt^2 \ .$$

We call $I\!I_t$ the second fundamental form of the hypersurface $\Sigma \times \{t\}$ for g, and $I\!I\!I_t$ the corresponding third fundamental form. Choose $m \in \Sigma$, and $x, y \in T_m\Sigma$. Then a classical computation (which was done, in a slightly more general case, in the proof of Lemma 1) shows that:

$$\frac{dI\!I_t}{dt}(x,y) = -I\!I\!I_t(x,y) + \langle R(x,n)y, n\rangle .$$

Let (e_i) be an orthonormal frame at m. Call R the Riemann curvature tensor of g, and R_t the curvature tensor of k_t. By the Gauss formula, for $i \in \mathbf{N}$:

$$\langle R_t(x, e_i)y, e_i\rangle = \langle R(x, e_i)y, e_i\rangle + (I\!I_t \owedge I\!I_t)(x, e_i, y, e_i) ,$$

where $\owedge$ is the Kulkarni-Nomizu product (see [Bes87]). Taking the trace and calling ric the Ricci curvature of M and ric_t the Ricci curvature of k_t leads to:

$$\text{ric}_t(x,y) = \text{ric}(x,y) - \langle R(x,n)y, n\rangle + \sum_i (I\!I_t \owedge I\!I_t)(x, e_i, y, e_i)$$

and

$$\sum_{i=1}^{m} (I\!I_t \owedge I\!I_t)(x, e_i, y, e_i) = H_t I\!I_t(x,y) - I\!I\!I_t(x,y)$$

so that

$$\frac{dI\!I_t}{dt} = \text{ric} - \text{ric}_t + H_t I\!I_t - 2 I\!I\!I_t .$$

Now $I\!I_t = -dk_t/dt$, and ric_t can be considered as a second-order elliptic operator in k_t. So k_t satisfies

$$\frac{d^2 k_t}{dt^2} = P(k_t) + R\left(\frac{k_t}{dt}\right) \tag{12}$$

where P is a second-order elliptic operator and R is an operator of degree 0 (all solutions of eq. (12) do not correspond to germs of Einstein manifolds around an umbilic surface; for instance, for $S = 0$ and $m = 2$, only constant curvature metrics can be induced on umbilic surfaces in Euclidean space).

Now we can apply the Cauchy-Kowalevskaya theorem (or the Cartan-Kähler theorem) which shows that eq. (12) has a unique analytic solution. On the other hand, Corollary 2 asserts that Σ has to be analytic in M, and then k_t has to be analytic, and also g (see [Bes87], 5.F). Equation (12) therefore has a unique solution, which is analytic. $\square$

Note that the solutions of equation (12) might not correspond to a germ of Einstein manifold around Σ, but such a germ is always obtained as a solution of eq. (12), and so is unique.

The next step is an inequality concerning the integral of the mean curvature squared.

Proposition 2 *Let (M, g) be an Einstein manifold with boundary, with scalar curvature S. Call $\overline{S}$ the scalar curvature of $(\partial M, g_{|\partial M})$. Then the mean curvature $H = \text{tr}(I\!I)$ of ∂M satisfies*

$$\frac{\overline{S}}{m-1} - \frac{S}{m+1} \le \frac{H^2}{m} ,$$

with equality if and only if ∂M is umbilic.

Proof: Call $\overline{\text{ric}}$ the Ricci curvature of $(\partial M, g_{|\partial M})$, and $K(x,y)$ (resp. $\overline{K}(x,y)$) the sectional curvature of g (resp. $g_{|\partial M}$) on the 2-plane generated by x, y. Let $(u_1, \cdots, u_n)$ be an orthonormal frame on ∂M for which $I\!I$ is diagonal, with eigenvalues $k_1, \cdots, k_n$. Then, by the Gauss formula:

$$\overline{K}(u_i, u_j) = K(u_i, u_j) + k_i k_j$$

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and, taking the trace:

$$\overline{\mathrm{ric}}(u_i, u_i) = \mathrm{ric}(u_i, u_i) - K(u_i, n) + k_i \left(\sum_{j \neq i} k_j \right) .$$

Taking the trace once more:

$$\overline{S} = S - 2\mathrm{ric}(n,n) + \sum_i k_i(H - k_i) = S - \frac{2S}{m+1} + H^2 - \sum_i k_i^2 . \tag{13}$$

Now

$$H^2 = \left(\sum_i 1.k_i \right)^2 \leq \left(\sum_i 1^2 \right) \left(\sum_i k_i^2 \right) = m \sum_i k_i^2$$

with equality if and only if all k_i are equal. The result follows. $\square$

This inequality could be interesting in itself. For instance, if $S = -m(m+1)$ and H is bounded above by some constant, it implies that the scalar curvature of ∂M is negative, which has some topological consequences.

By the way, this computation also leads to the following partial extension of Theorem 6:

Remark 3 *The second mean curvature $H_2 = \sum_{i<j} k_i k_j$ of the boundary of an Einstein manifold is:*

$$2H_2 = \overline{S} - \frac{m-1}{m+1} S$$

Therefore, it is pointwise constant in an Einstein variation of the metric which vanishes on the boundary.

Proof: It follows from eq. (13). $\square$

Now from Proposition 2 and Lemma 8 we get:

Corollary 4 *Let $(M, \partial M)$ be a compact manifold with convex (or concave) boundary. Let $(g_t)_{t \in [0,1]}$ be a one-parameter family of Einstein metrics on M with scalar curvature S, such that ∂M is umbilic for g_0, and that the metric induced by g_t on ∂M is constant. Call $\mathcal{H}_t$ the integral of the mean curvature of H for g_t. Then:*

1. *if $S > 0$ and $H > 0$ (resp. $H < 0$) on each connected component of ∂M, then both $\mathcal{H}_t$ and V_t are minimal (resp. maximal) for t_0, and the variation H_t' of H_t vanishes for $t = 0$;*

2. *if $S < 0$ and $H > 0$ (resp. $H < 0$) on each connected component of ∂M, then $\mathcal{H}_t$ is minimal and V_t is maximal (resp. $\mathcal{H}_t$ is maximal and V_t is minimal) for t_0, and the variation H_t' of H_t vanishes for $t = 0$;*

3. *if $S = 0$, then ∂M is umbilic for all $t \in [0,1]$, i.e. its second fundamental form does not change.*

Proof: By Proposition 2, H^2 is pointwise minimal over ∂M when ∂M is umbilic. If, for instance, $H > 0$ on each connected component of ∂M, this shows that $\mathcal{H}_t$ is also minimal for $t = 0$. By Corollary 9, V is also minimal for $t = 0$ when $S > 0$, and maximal when $S < 0$. This proves assertions 1 and 2.

For assertion 3, the integral of H over ∂M is constant by Corollary 9, while H^2 is pointwise minimal over ∂M. Therefore, H^2 has to be constant. By the equality case in Proposition 2, ∂M has to remain umbilic in that case. $\square$

The Corollaries 3 and 4 lead to a description of the non-rigid Ricci-flat manifolds with umbilic boundary.

Lemma 1 *Suppose $(M, \partial M)$ is a compact $(m+1)$-manifold with boundary, and $(h_t)_{t \in [0,1]}$ is a non-trivial 1-parameter family of Ricci-flat metrics on M inducing the same metric on ∂M, and such that ∂M is umbilic for h_0. Then ∂M has at least 2 connected components, and (h_t) corresponds to the displacement of some connected component(s) of ∂M under the flow of some Killing field(s) of M.*

Note that this is rather restrictive, since a "generic" Einstein manifold with boundary should not admit any Killing field. Some examples are given bellow.

Proof: By Corollary 4, h does not change the induced metric or the second fundamental form of ∂M. Call $\Sigma_1, \cdots, \Sigma_N$ the connected components of ∂M. Then, by Corollary 3, each connected component $\partial_i M$ of ∂M has a neighborhood $\Omega_{i,t}$ which does not change. Therefore, the deformation (h_t) corresponds to the displacement of some $\partial_i M$ under Killing fields on M. $\square$

We shall give some examples of what can happen; this is easier using the following elementary:

Proposition 3 *Let (N, g_0) be an Einstein m-manifold with scalar curvature $m(m-1)k$, consider the product $M \times \mathbf{R}$ with the warped metric*

$$g := dt^2 + f(t)^2 g_0$$

where f is a function defined on some interval $I \subset \mathbf{R}$. Then g is Einstein with scalar curvature $m(m+1)k'$ if and only if $f''(t) = -k'f(t)$ and $k = k'f(t)^2 + f'(t)^2$ for all t. Then each hypersurface $N \times \{t\}$ is umbilic in M.

Proof: First check that the Levi-Civita connection D of g is related to the Levi-Civita connection $\overline{D}$ of g_0 by the following formulas: if n is the unit normal vector to $N \times \{t\}$, and x, y are vector fields on N (and their extensions on M) then

$$D_x y = \overline{D}_x y - \frac{f'}{f} g(x, y) n$$

$$D_n x = D_x n = \frac{f'}{f} x$$

$$D_n n = 0$$

This is because those expressions define a torsion-free connection compatible with g.

Now this expressions of D shows that each hypersurface $N \times \{t_0\}$ in M is umbilic, with second fundamental form $-(f'/f)g$. Therefore, the Gauss formula shows that the sectional curvature of M on any 2-plane tangent to $N \times \{t_0\}$ is:

$$K_M = \frac{1}{f^2} K_N - \frac{f'^2}{f^2}$$

and on each 2-plane containing the direction normal to $N \times \{t_0\}$:

$$K_M = -\frac{f''}{f}$$

which shows that $\mathrm{ric}^M(n, n) = -mf''/f$, and leads to the first condition. Taking a trace, we see that for x tangent to $N \times \{t\}$:

$$\mathrm{ric}^M(x, x) = \frac{k(m-1) - (m-1)f'^2 + ff''}{f^2} g(x, x)$$

and the second condition follows. $\square$

Since we are interested in Ricci-flat metrics, we have to take $k' = 0$, and we can use this proposition in two ways: either $k = 0$ and $f(t) = 1$, or $k = 1$ and $f(t) = t$.

The simplest example is the case when g_0 is the canonical metric on the sphere S^m:

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Example 1 *Consider the unit ball $B^{m+1} \subset \mathbf{R}^{m+1}$. Any 1-parameter Einstein deformation of the metric in B^{m+1} which doesn't change the induced metric on the boundary S^m is trivial.*

On the other hand:

Example 2 *Consider the "cylinder" $\Omega := T^m \times [0,1]$. There exists a 1-parameter family of deformations of the metric on Ω which does not change the induced metric on the boundary.*

This deformation is obtained by one of the boundary components along the axis of the cylinder. This happens because there is a Killing vector field, which is parallel to the axis.

4   Codimension one foliations

We give in this section some simple formulas obtained by applying Theorem 1 to codimension one foliations of Einstein manifolds. Let $(\Sigma_t)_{t \in I}$ be a smooth one-parameter family of hypersurfaces in an Einstein $(m+1)$-manifold with scalar curvature $m(m+1)K$. Suppose that the Σ_t define a foliation of a domain $\Omega \subset M$. For each $x \in \Omega$, let $H_2(x)$ be the second mean curvature of Σ_t at x (for t such that $x \in \Sigma_t$) and let $S_\Sigma(x)$ be the scalar curvature of Σ_t for the induced metric. Then:

Theorem 10 *The volume $V(\Omega)$ of Ω is:*

$$mKV(\Omega) = 2 \int_\Omega H_2 dV + \int_{\partial\Omega} H(\partial\Omega)dA \ , \tag{14}$$

$$mKV(\Omega) = \int_\Omega H_\Sigma^2 - tr(I\!I\!I_\Sigma)dV + \int_{\partial\Omega} H(\partial\Omega)dA \ , \tag{15}$$

and also

$$m^2 KV(\Omega) = \int_\Omega S_\Sigma dV + \int_{\partial\Omega} H(\partial\Omega)dA \ . \tag{16}$$

Here $H(\partial\Omega)$ is the mean curvature of $\partial\Omega$. H_Σ, S_Σ and $I\!I\!I_\Sigma$ are the mean curvature, sectional curvature and third fundamental form of Σ_t.

Proof: Suppose for instance that $I = [0,1]$, and note:

$$V_t = \mathrm{Vol}\left(\cup_{s=0}^t \Sigma_s\right) \ .$$

Denote again the variations of V_t, I_t and H_t by a prime. Choose a parametrization $\phi_t : \Sigma \to M$ such that $\Sigma_t = \phi_t(\Sigma)$, and let $\phi_t' = v + fn$, where $v \in T\Sigma$ and n is the unit normal vector to Σ_t. Then, by eq. (2):

$$
\begin{aligned}
nKV_t' &= \int_{\Sigma_t} H_t' + \frac{1}{2}\langle I_t', I\!I_t \rangle dA \\
&= \frac{d}{dt}\int_{\Sigma_t} H_t dA - \int_{\Sigma_t} H_t dA' + \int_{\Sigma_t} \frac{1}{2}\langle I_t', I\!I_t \rangle dA \\
&= \frac{d}{dt}\int_{\Sigma_t} H_t dA + \int_{\Sigma_t} \frac{1}{2}\langle I_t', I\!I_t \rangle - H_t(-fH_t + \overline{\mathrm{div}}(v))dA \\
&= \frac{d}{dt}\int_{\Sigma_t} H_t dA + \int_{\Sigma_t} -f\langle I\!I_t, I\!I_t \rangle + \langle \overline{\delta}^* v, I\!I_t \rangle + fH_t^2 - H_t\overline{\mathrm{div}}(v))dA \\
&= \frac{d}{dt}\int_{\Sigma_t} H_t dA + \int_{\Sigma_t} 2fH_2 + dH_t(v)dA
\end{aligned}
$$

and eq. (14) follows, because $\overline{\delta}I\!I_t = -dH_t$ (see (3)). Eq. (15) is a direct consequence because $H^2 = \mathrm{tr}(I\!I\!I) + 2H_2$.

Taking twice the trace of the Gauss equation for Σ_t shows that:

$$S_\Sigma = m(m-1)K + 2H_2 \ ,$$

so that eq. (14) becomes:

$$mKV(\Omega) = \int_\Omega S_\Sigma dV - m(m-1)KV(\Omega) + \int_{\partial\Omega} HdA$$

which proves (16). $\square$

This leads for instance to the following simple consequence:

Corollary 5 *If $K > 0$, then no open domain of M has a foliation by closed, minimal hypersurfaces. If $K = 0$, any such foliation is by totally geodesic hypersurfaces.*

Proof: Suppose first that $K > 0$. Apply eq. (15) to such a foliation. The boundary term vanishes, and the right-hand side is therefore non-positive, while the left-hand side is positive, a contradiction.

If $K = 0$, the same argument shows that $H_2 \equiv 0$, therefore $I\!I\!I \equiv 0$ on each hypersurface by eq. (15), and each hypersurface is totally geodesic. $\square$

This strongly contrasts with the negatively curved case; for instance, it is conjectured that a hyperbolic 3-manifold which fibers over the circle admits a foliation by compact minimal surfaces. Equation (15) indicates that such a minimal foliation should have a remarkable property: the Gauss curvature of each leaf, integrated against a weight corresponding to the amplitude of the normal deformation, should be constant.

Corollary 5 is not too difficult to obtain by other methods; it is interesting to remark, however, that equations (14) and (15) can also be used to obtain more general results, for instance to give an integral lower bound on the mean curvature of a foliation by minimal hypersurfaces in a positively curved Einstein manifold.

5   A quick tour of integral geometry

In this section, we give a summary of some concepts of integral geometry which permit us to interpret some of our results more geometrically. Of course, it cannot be hoped that we can give anything resembling a comprehensive survey. A reader more interested in this fascinating subject is referred to the treatises of Santaló [San76] and of Burago and Zalgaller [BuZ].

First, we recall some formulas of Crofton type. Consider n-dimensional Euclidean space E^n and consider the Grassmanian of all the affine m-planes in $E^n - G_m^n$. This has a measure invariant with respect to the isometry group of E^n. Any two such measures differ by a constant factor. There is a standard way to normalize, which will be implicit in the identities we shall state. Anyway, call the "canonical" invariant volume form dv_m^n. There is a natural functional defined on (not necessarily) convex sets K in E^n, to wit,

$$P_m(K) = \int_{L_m \cap K \neq \emptyset} dv_m^n.$$

Another natural functional is the following: Consider the space of m-dimensional linear subspaces of E^n, and consider the average m-dimensional area of the projections of K onto such subspaces. This is the so-called *Quermassintegral* $W_m(K)$. It is fairly clear that P_m and W_m are related, and indeed, one of the fundamental formulas of integral geometry ([San76, eq. (14.1)]) is

$$P_m(K) = \frac{nO_{n-2}O_{n-3}\cdots O_{n-m-1}}{(n-m)O_{m-1}\cdots O_1 O_0}W_m(K), \tag{17}$$

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where O_i is just the surface area of the i-dimensional unit sphere S^i (it should be noted that the fraction in eq. (17) is just the volume of the Grassmanian of the m-dimensional subspaces in E^n).

Equation (17) allows us to relate the quantity $P_m(K)$ to the integral of the m-th symmetric function of curvature, as follows: Consider the volume V_ϵ of the ϵ neighborhood of K. A simple computation with radii of curvature shows that if $\kappa = \kappa_1, \ldots, \kappa_{n-1}$ is the vector of principal curvatures of ∂K (we assume that ∂K is at least C^2 smooth), and $\sigma_m(\kappa)$ is the m-th symmetric function of curvature, then

$$V_\epsilon(K) = V(K) + \sum_{m=0}^{n-1} \frac{\epsilon^{m+1}}{m+1} \int_{\partial K} \sigma_m(K).$$ (18)

On the other hand, there is another expression for $V_\epsilon(K)$, due to Steiner (see [San76, III.13.3]):

$$V_\epsilon(K) = \sum_{i=0}^{n} \binom{n}{i} W_i(K) \epsilon^i.$$ (19)

Comparing the coefficients of the powers of ϵ in formulas (18) and (19), we see that:

$$m \int_{\partial K} \sigma_{m-1}(K) ds = \binom{n}{m} W_m(K).$$

In other words, the integral mean curvatures of K are directly expressible in terms of the measures of the set of planes intersecting K, and the average projection measures. In particular, since σ_0 is equal to 1, we see that the area of ∂K is equal to a constant factor times the measure of the set of lines intersecting K, while the total (first) mean curvature is a constant times the measure of the set of 2-planes intersecting K. Since we are especially interested in $n = 3$, we will write down the constants explicitly in that case:

$A(\partial K) = 3W_1(K)$, while $P_1(K) = \frac{6\pi}{4} W_1(K)$, so

$$A(\partial K) = \frac{2}{\pi} P_1(K).$$ (20)

On the other hand, $\int_{\partial K} (k_1 + k_2) = 3W_2(K)$, while $P_2(K) = \frac{3}{2} W_2(K)$, so

$$\int_{\partial K} (k_1 + k_2) = 2P_2(K).$$ (21)

So far, we have talked about convex bodies in a Euclidean setting, but the theory can be extended to other symmetric spaces, in particular, to H^n and S^n. The expressions become a bit more complicated in general, but in three dimensions, they are simple enough; the following formulas are in [San76, eq. (17.62)]:

$$P_1(K) = \frac{\pi}{2} A(\partial K), \qquad P_2(K) = \frac{1}{2} \int_{\partial K} (k_1 + k_2) + kV(K),$$ (22)

where k is the sectional curvature of the ambient space (so the formula reduces to eq. (21) when $k = 0$). It should be noted that there is yet another interpretation of the quantity $P_2(K)$, in terms of the *polar map* of a spherical or hyperbolic convex body (the spherical version is classical, the hyperbolic has been studied in the first author's thesis [RivH93]): this map associates to K the set K^* of hyperplanes intersecting K. For the sphere S^n, the set K^* can be naturally viewed as a convex body in S^n, for H^n the set of hyperplanes can be naturally viewed as the de Sitter space S_1^{n-1}

Alexandrov's inequality ([BuZ, p. 145]) is the following: for a convex K in E^n, the cross-sectional measures satisfy:

$$V_j^i(K) \geq v_n^{i-j} V_i^j(K), \qquad j \geq i,$$ (23)

with equality if and only if K is a ball in R^n. Here V_i can be defined by the following formula: ([BuZ, p. 140]):

$$V_k = \frac{1}{n\binom{n-1}{m}} \int_{\partial K} \sum_j k_{j_1} \ldots k_{j_{n-m-1}} dF(x),$$

where v_n is the volume of the n-dimensional unit ball ; k_i $(i = 1, \ldots, n-1)$ are the principal curvatures of ∂K at the point x of ∂K, dF is the area element of ∂K, the sum is taken over all possible finite sequence of indices $j_1, \ldots, j_{n-m-1}$. In particular, $V_{n-2}(K)$ is related to $Q(S) = \int_S H$ by $Q(S) = n(n-1)V_{n-2}(K)$. Aleksandrov's inequality in E^3 can thus be restated as follows : Among all convex bodies with a fixed P_1, the ball has the biggest P_2. Our Theorem 12 is exactly the extension of this result to other three-dimensional space forms.

6 Extending the Aleksandrov inequality

This section contains applications of the previous results in the simple setting of three-dimensional space-forms, esp. H^3. Thus we consider a smooth, strictly convex surface Σ in a constant curvature space M which might be $S^3, \mathbf{R}^3, H^3$ or S_1^3 (in this case we suppose that Σ is space-like).

To keep notations close to that of the previous section, we define a functional P_2 as:

$$2P_2 := \int_\Sigma H da - 2\epsilon K_0 V ,$$

where K_0 is the sectional curvature of M, and $\epsilon = 1$ if M is Riemannian, $\epsilon = -1$ if $M = S_1^3$.

Note that, for any deformation of Σ:

$$\left(\int_\Sigma H da\right)' = \int_\Sigma H' da + \frac{1}{2}\int_\Sigma H\langle I', I\rangle da .$$

Therefore, as a consequence of equations (2) and (5), we have for any deformation of Σ:

$$P_2' = -\frac{1}{4}\int_\Sigma \langle I', I\!\!I - HI\rangle da \tag{24}$$

Consider a normal deformation of Σ, i.e. an infinitesimal deformation by a vector field fn, where n is the (exterior) unit normal to Σ. Then $I' = 2f I\!\!I$, so that:

$$2P_2' = \int_\Sigma \langle -f I\!\!I, I\!\!I - HI\rangle da = \int_\Sigma 2f K_e da \tag{25}$$

where $K_e := \det(I\!\!I)$ is the extrinsic curvature of Σ. On the other hand, the area of Σ varies as:

$$A' = \frac{1}{2}\int_\Sigma \langle 2f I\!\!I, I\rangle da = \int_\Sigma f H da \tag{26}$$

As a consequence, we already find an extremely simple proof of a result with a flavor of classical differential geometry. It can be seen as a consequence of some more general results (see [Ros88], and also [EH89]) but we include it here because of its extremely simple proof.

Theorem 11 *Suppose that Σ is a smooth, strictly convex surface in a 3-dimensional space-form, and that there exists a constant $k \in \mathbf{R}_+$ such that, on Σ, $K_e = kH$. Then Σ is totally umbilical.*

Proof: Suppose that $K_e = kH$. Then, by equations (26) and (25), Σ is a critical point of P_2 among surfaces with the same area. But it is well known (see [Pog73]) that all variations I' of I are induced by deformations of Σ. Therefore, eq. (24) shows that there exists a constant k' such that $I\!\!I - HI = k'I$, so that Σ is totally umbilical. $\square$

We now turn to the extension of the classical Alexandrov inequality (see Section 5) for convex surfaces in Euclidean space to three-dimensional space-forms.

Theorem 12 *Let S be a compact convex surface in H^3 (resp. R^3, S^3). Let $V(S)$ be the volume of the interior of S, and call $2P_2(S) := \int_S H da + 2V(S)$ (resp. $2P_2(S) = \int_S H da$, $2P_2(S) = \int_S H da - 2V(S)$). There exists a (unique modulo global isometries) umbilical surface S_0 with $Area(S_0) = Area(S)$, and $P_2(S_0) \geq P_2(S)$.*

Remark 4 *An elementary consequence is the well-known fact that a convex surface in S^3 has area at most 2π.*

Proof. We will give the proof of Theorem 12 in the hyperbolic case, the other two situations are very similar. For $k_0 > 0$, let $\mathcal{C}_{A,k_0}$ be the space of smooth, convex surfaces in H^3 with area A and principal curvatures at most k_0, and containing a given fixed point x_0. It is again an elementary consequence of eq. (24) and of [Pog73] that the only critical points of P_2 in $\mathcal{C}_{A,k_0}$ are the umbilical hypersurfaces (there is a unique such surface in $\mathcal{C}_{A,k_0}$ modulo the global isometries of H^3).

It is therefore sufficient to prove that P_2 has a maximum in the interior of $\mathcal{C}_{A,k_0}$. This will be a consequence of the following points:

1. if $S \in \mathcal{C}_{A,k_0}$ contains a point s where a principal curvature vanishes, then there exists an infinitesimal deformation of S increasing the minimum of the principal curvatures and increasing P_2, while leaving the area constant;

2. if $k_0 > 1$ and if $S \in \mathcal{C}_{A,k_0}$ contains a point s where a principal curvature is equal to k_0, then there exists a deformation of S decreasing the maximum of the principal curvatures and increasing P_2;

3. for each $M > 0$, there exists $L > 0$ such that if $S \in \mathcal{C}_{A,k_0}$ has (extrinsic) diameter above L, then $P_2(S) \leq -M$.

Theorem 12 follows, because a maximizing sequence for P_2 can neither "degenerate" (because of point (3)), nor converge to a surface with a vanishing principal curvature (point (1)) or a principal curvature equal to k_0 (point (2)).

To prove point (1), note that the equation (4) simplifies here to:

$$I\!I' = H_f + fI - f I\!I\!I \tag{27}$$

Therefore, to insure that a deformation fn increases the minimum of the principal curvatures at a point where this minimum vanishes, it is enough to have: $f \geq \|H_f\|$ over S. But, if a principal curvature of S vanishes, then S is not umbilical, so that H and K_e are not proportional on S. It is then easy to check using equations (25) and (26) that there exists a normal deformation of S with the right properties.

Point (2) can be proved in the same way, the condition on f is now that $(k_0 - 1)f \geq \|H_f\|$.

Finally, for point (3), let S be a convex surface in H^3, and, for $\epsilon \geq 0$, call E_ϵ the set of points at distance at most ϵ from the interior of S, and $S_\epsilon := \partial E_\epsilon$. Let $\mathcal{H}_\epsilon$ be the integral mean curvature of S_ϵ, and let $\mathcal{A}_\epsilon$ be its area and $\mathcal{V}_\epsilon$ the volume of its interior E_ϵ. Equation (2) shows that:

$$-2\frac{d\mathcal{V}_\epsilon}{d\epsilon} = \frac{d\mathcal{H}_\epsilon}{d\epsilon} + \int_{S_\epsilon} \langle I\!I, I\!I - HI \rangle da ,$$

so that:

$$2\frac{d\mathcal{V}_\epsilon}{d\epsilon} + \frac{d\mathcal{H}_\epsilon}{d\epsilon} = \int_{S_\epsilon} K_e da = \int_{S_\epsilon} (K+1)da = 4\pi + \mathcal{A}_\epsilon ,$$

where we have used the Gauss-Bonnet theorem. Since $d\mathcal{V}_\epsilon/d\epsilon = \mathcal{A}_\epsilon$, we have:

$$\frac{d\mathcal{V}_\epsilon}{d\epsilon} + \frac{d\mathcal{H}_\epsilon}{d\epsilon} = 4\pi ,$$

and therefore:

$$\frac{d^3\mathcal{V}_\epsilon}{d\epsilon^3} + \frac{d\mathcal{V}_\epsilon}{d\epsilon} = 4\pi \ .$$

Integrating this EDO leads to a classical formula for $\mathcal{V}_\epsilon$:

$$\mathcal{V}_\epsilon = \mathcal{A}_0 \sinh(\epsilon) + 4\pi(\epsilon - \sinh(\epsilon)) + \mathcal{H}_0(\cosh(\epsilon) - 1) + \mathcal{V}_0 \tag{28}$$

Now suppose that S has extrinsic diameter at least L, then E_0 contains a segment γ of length at least L. Applying equation (28) to a sequence of convex surfaces in E_0 which converges to γ, we find a lower bound for $\mathcal{V}_\epsilon$:

$$\mathcal{V}_\epsilon \geq 4\pi(\epsilon - \sinh(\epsilon)) + 2\pi L(\cosh(\epsilon) - 1)$$

so that, for any $\epsilon \geq 0$:

$$\mathcal{A}_0 \sinh(\epsilon) + \mathcal{H}_0(\cosh(\epsilon) - 1) + \mathcal{V}_0 \geq 2\pi L(\cosh(\epsilon) - 1)$$

Taking the limit as $\epsilon \to \infty$ shows that:

$$\mathcal{A}_0 + \mathcal{H}_0 \geq 2\pi L$$

and point (3) follows.

This finishes the proof of Theorem 12. □

Note that the "local" part of this argument extends partly to higher dimensions, again for Einstein manifolds with convex boundaries. Namely, if $(M, \partial M)$ is such a manifold, we say that it is "rigid" if it admits no infinitesimal Einstein deformation which does not change the induced metric on the boundary. It is proved in [Sch98] that, in that case, all infinitesimal deformations of the metric on ∂M are induced (uniquely) by Einstein deformations of M. Therefore, it is still true in that case that, if $(M, \partial M)$ is a critical point of P_2 among Einstein metrics with the same "area", then it is umbilical. We do not know whether there exists any non-rigid Einstein metric with negative curvature and strictly convex boundary.

Acknowledgements

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